

GRAPHS OF 2-TORUS ACTIONS

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ABSTRACT. It has been known that an effective smooth $(\mathbb{Z}_2)^k$ -action on a smooth connected closed manifold M^n fixing a finite set can be associated to a $(\mathbb{Z}_2)^k$ -colored regular graph. In this paper, we consider abstract graphs (Γ, α) of $(\mathbb{Z}_2)^k$ -actions, called abstract 1-skeletons. We study when an abstract 1-skeleton is a colored graph of some $(\mathbb{Z}_2)^k$ -action. We also study the existence of faces of an abstract 1-skeleton (note that faces often have certain geometric meanings if an abstract 1-skeleton is a colored graph of some $(\mathbb{Z}_2)^k$ -action).

1. INTRODUCTION

Goresky, Kottwitz and MacPherson in [GKM] showed that there is an essential connection between torus actions and regular graphs, i.e., there is a large class of T -manifolds (called GKM manifolds), each M of which can be associated to a unique regular graph Γ_M so that the equivariant cohomology of M can be computed by the associated regular graph Γ_M , where T is a torus. A series of works by Guillemin and Zara further showed that more topological and geometrical properties of M can be read out from Γ_M (see, [GZ1]-[GZ4]).

It was shown in [L] that the above idea can be extended to 2-torus actions, leading to study the equivariant cobordism classification and the Smith problem for 2-torus actions. Specifically, assume that (Φ, M) is an effective smooth G -action on a smooth connected closed manifold M fixing a finite set (note that here (Φ, M) has less restriction than a GKM manifold), where $G = (\mathbb{Z}_2)^k$, a 2-torus of rank k . Then we know from [L, Section 2] that the action (Φ, M) defines a regular graph $\Gamma_{(\Phi, M)}$ with the vertex set M^G . This graph is equipped with a natural map (or a G -coloring) α from the set $E_{\Gamma_{(\Phi, M)}}$ of all edges of $\Gamma_{(\Phi, M)}$ to all non-trivial elements of $\text{Hom}(G, \mathbb{Z}_2)$, satisfying the following properties:

- (P1) For each vertex p of $\Gamma_{(\Phi, M)}$, the image set $\alpha(E_p)$ spans $\text{Hom}(G, \mathbb{Z}_2)$, where E_p denotes the set of all edges adjacent to p .
- (P2) For each edge e of $\Gamma_{(\Phi, M)}$,

$$\alpha(E_p) \equiv \alpha(E_q) \pmod{\alpha(e)}$$

where p and q are two endpoints of e .

The pair $(\Gamma_{(\Phi, M)}, \alpha)$ is called a G -colored graph of (Φ, M) . $(\Gamma_{(\Phi, M)}, \alpha)$ doesn't only contain the most essential equivariant cobordism information of (Φ, M) , but it also indicates the relationship among G -representations on tangent spaces at fixed points,

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so that it can be used to study the equivariant cobordism classification and the Smith problem of (Φ, M) . Actually, from the colored graph $(\Gamma_{(\Phi, M)}, \alpha)$ we can read out all G -representations on tangent spaces at fixed points, which exactly consist of all images $\alpha(E_p), p \in M^G$. In particular, $\{\alpha(E_p) | p \in M^G\}$ determines a complete equivariant cobordism invariant $\mathcal{P}_{(\Phi, M)}$ of (Φ, M) , where $\mathcal{P}_{(\Phi, M)}$ is obtained by deleting same pairs in $\{\alpha(E_p) | p \in M^G\}$.

In this paper, we shall be concerned with an abstract pair (Γ, α) , where Γ is a finite n -valent regular graph without loops (i.e., edges with only an endpoint), and α satisfies (P1) and (P2). We call (Γ, α) an *abstract 1-skeleton of type (n, k)* (cf [GZ2, Definition 2.1.1]). An easy observation shows that generally (Γ, α) may fail to be a colored graph induced by some G -action (Φ, M) (see also Example 1 of Section 2). A natural question is

(Q1) *When is (Γ, α) a G -colored graph of some G -action (Φ, M) ?*

When $k = 1$, we give a complete answer for (Q1), and when $k \geq 2$, we give a partial answer for (Q1). We show that when $k \geq 2$, if for each vertex $p \in V_\Gamma$, all vectors of $\alpha(E_p)$ are pair-wise independent in $\text{Hom}(G, \mathbb{Z}_2)$, then $\{\alpha(E_p) | p \in V_\Gamma\}$ must be the fixed point data of some G -action (Φ, M) . An example shows that the restriction of the pair-wise independence of α is necessary in the general case. However, we cannot make sure that Γ is just a graph $\Gamma_{(\Phi, M)}$ of (Φ, M) .

We also consider those connected regular subgraphs of (Γ, α) , each of which, with the restriction of α to it, is still an abstract 1-skeleton on its own right. We shall call those regular subgraphs with restrictions of α the *faces* of (Γ, α) . Generally, each face has its geometric meaning if (Γ, α) is a G -colored graph of a G -action (Φ, M) . Actually it is often a H -colored graph of H -action on N , where H is a subtorus of G , and N is a component of the fixed point set of G/H acting on M . We shall consider the existence of faces of an abstract 1-skeleton. Specifically, the following problem will be studied:

(Q2) *Let (Γ, α) be an abstract graph of type (k, n) . Assume given a vertex $p \in V_\Gamma$ and a set of $m (\geq 2)$ edges in E_p . Is there always a unique face containing these m edges?*

The l -independence of α for (Q2) is essentially important. We shall show that if α is l -independent, then any m edges in E_p with $m < l$ extend to a unique face. We shall also study the intersection property of faces, and then use it to consider the n -connectedness of Γ , obtaining a sufficient condition that Γ is n -connected.

In the extreme case, an abstract 1-skeleton of type (n, n) has its own special properties. For example, the answer of (Q2) in this case is always *yes*, so that each such abstract graph (Γ, α) can be associated with a unique simplicial poset, and so (Γ, α) has a geometric realization $|\Gamma, \alpha|$. If (Γ, α) is a colored graph of a small cover over a simple convex polytope P , then $|\Gamma, \alpha|$ is exactly the boundary of P . However, generally $|\Gamma, \alpha|$ even is not a closed manifold. In [BL], Bao and Lü introduced the method of the skeletal expansion and gave a detailed investigation on $|\Gamma, \alpha|$.

The paper is organized as follows. In Section 2, we introduce the notions of an abstract 1-skeleton and the l -independence of α , and study the question (Q1). The

notion of a face of (Γ, α) is given in Section 3, and then the question (Q2) is discussed. In Section 4, we consider the abstract 1-skeletons of type (n, n) .

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2. ABSTRACT 1-SKELETONS

Throughout the following one assumes that $G = (\mathbb{Z}_2)^k$ with $k \geq 1$.

2.1. G -representations. Following [CF, Section 31], let $R_n(G)$ denote the set generated by the representation classes of dimension n , which naturally forms a vector space over \mathbb{Z}_2 . Then $R_*(G) = \sum_{n \geq 0} R_n(G)$ is a graded commutative algebra over \mathbb{Z}_2 with unit. The multiplication in $R_*(G)$ is given by $[V_1] \cdot [V_2] = [V_1 \oplus V_2]$. Let $\text{Hom}(G, \mathbb{Z}_2)$ be the set of all homomorphisms $\rho : G \rightarrow \mathbb{Z}_2 = \{\pm 1\}$, which consists of 2^k distinct homomorphisms, and let ρ_0 denote the trivial element in $\text{Hom}(G, \mathbb{Z}_2)$, i.e., $\rho_0(g) = 1$ for all $g \in G$. Every irreducible real representation of G is one-dimensional and has the form $\lambda_\rho : G \times \mathbb{R} \rightarrow \mathbb{R}$ with $\lambda_\rho(t, r) = \rho(t) \cdot r$ for some $\rho \in \text{Hom}(G, \mathbb{Z}_2)$. Obviously there is a 1-1 correspondence between all irreducible real representations of G and all elements of $\text{Hom}(G, \mathbb{Z}_2)$. $\text{Hom}(G, \mathbb{Z}_2)$ forms an abelian group with addition given by $(\rho + \sigma)(g) = \rho(g) \cdot \sigma(g)$, so it is also a vector space over \mathbb{Z}_2 with standard basis $\{\rho_1, \dots, \rho_k\}$ where ρ_i is defined by mapping $g = (g_1, \dots, g_i, \dots, g_k)$ to g_i . Thus, we can identify $R_*(G)$ with the graded polynomial algebra over \mathbb{Z}_2 generated by $\text{Hom}(G, \mathbb{Z}_2)$. Namely, $R_*(G)$ is isomorphic to the graded polynomial algebra $\mathbb{Z}_2[\rho_1, \dots, \rho_k]$.

2.2. Abstract 1-skeletons. Let Γ be a finite regular graph of valence n without loops such that $n \geq k$. If there is a map $\alpha : E_\Gamma \rightarrow \text{Hom}(G, \mathbb{Z}_2) \setminus \{\rho_0\}$ such that

- (1) for each vertex $p \in V_\Gamma$, the image $\alpha(E_p)$ spans $\text{Hom}(G, \mathbb{Z}_2)$, and
- (2) for each edge $e = pq \in E_\Gamma$,

$$\alpha(E_p) \equiv \alpha(E_q) \pmod{\alpha(e)}$$

then the pair (Γ, α) is called an *abstract 1-skeleton of type (k, n)* .

For each vertex $p \in V_\Gamma$, if any l elements in $\alpha(E_p)$ are linearly independent in $\text{Hom}(G, \mathbb{Z}_2)$, then one says that α is *l -independent* (cf [GZ2, Definition 2.1.2]).

Remark 1. Since $\text{Hom}(G, \mathbb{Z}_2)$ contains $2^k - 1$ different nonzero elements, one has that if α is pairwise linearly independent (i.e., 2-independent), then $k \geq 2$ and the valence of Γ is at most $2^k - 1$.

2.3. Realization problem. Now let (Γ, α) be an abstract 1-skeleton of type (k, n) . We shall consider the question of whether (Γ, α) can be realized as a G -colored graph of some G -action (Φ, M^n) .

When $k = 1$, $\text{Hom}(G, \mathbb{Z}_2)$ has only a non-trivial element, so for each vertex $p \in V_\Gamma$, all elements of $\alpha(E_p)$ are same, and for any two $p, q \in V_\Gamma$, $\alpha(E_p) = \alpha(E_q)$.

Proposition 2.1. *Let (Γ, α) is an abstract 1-skeleton of type $(1, n)$. Then (Γ, α) is a G -colored graph of a G -action if and only if the number of vertices of Γ is even.*

Proof. If (Γ, α) is a G -colored graph of a G -action (i.e., an involution) (Φ, M) , then the number of fixed points of (Φ, M) is the same as that of vertices of Γ . We know from [CF, Theorem 25.1] that if an involution fixes a finite set, then the number of fixed points must be even. Thus, the number of vertices of Γ is even.

Conversely, suppose that the number of vertices of Γ is even. With no loss of generality, assume that Γ is connected. Given a G -action Φ on a connected closed manifold M fixing a finite fixed set, we know from [L, Section 2] that any n -valent connected regular graph with the vertex set M^G can be used as the colored graph of (Φ, M) since all tangent G -representations at fixed points are the same. So, in order to complete the proof, it suffices to show that for each positive integer l , there is always an involution fixing $2l$ isolated points. In fact, a sphere S^n always admits an involution fixing only two isolated points. Then the equivariant connected sum of l copies of such involution on S^n along their free orbits produces a new \mathbb{Z}_2 -action (Ψ, N) , fixing $2l$ isolated points with same tangent G -representation. Thus, if the number of vertices of Γ is even, then (Γ, α) is a G -colored graph of a G -action. \square

When $k \geq 2$, the problem becomes more complicated, but the 2-independence of α makes sure that $\{\alpha(E_p) | p \in V_\Gamma\}$ is the fixed point data of some G -action.

Proposition 2.2. *Let (Γ, α) is an abstract 1-skeleton of type (k, n) with $k \geq 2$. If α is 2-independent, then $\{\alpha(E_p) | p \in V_\Gamma\}$ is the fixed point data of some G -action.*

Proof. According to tom Dieck-Kosniowski-Stong localization theorem (see [D], [KS] or [L, Theorem 3.2]), it suffices to show that for any symmetric polynomial function $f(x_1, \dots, x_n)$ over \mathbb{Z}_2 ,

$$\sum_{p \in V_\Gamma} \frac{f(\alpha(E_p))}{\prod_{e \in E_p} \alpha(e)} \in \mathbb{Z}_2[\rho_1, \dots, \rho_k]$$

where $f(\alpha(E_p))$ means that x_1, \dots, x_n in $f(x_1, \dots, x_n)$ are replaced by all elements in $\alpha(E_p)$. The idea of the following proof is essentially due to Guillemin and Zara [GZ1, Theorem 2.2], but the proof is included here for local completeness.

If α is 2-independent, then for each vertex p , all elements of $\alpha(E_p)$ are distinct. Thus, taking the common denominator, $\sum_{p \in V_\Gamma} \frac{f(\alpha(E_p))}{\prod_{e \in E_p} \alpha(e)}$ becomes

$$\frac{h}{\beta_1 \beta_2 \cdots \beta_u}$$

where $\beta_1, \beta_2, \dots, \beta_u \in \text{Hom}(G, \mathbb{Z}_2)$ are distinct. Now we want to show that h is actually divisible by each β_i . With no loss of generality, it suffices to prove that h can be divided by β_1 . Let $V_1 = \{p \in V_\Gamma | \beta_1 \in \alpha(E_p)\}$. It is easy to see that V_1 contains an even number of vertices of Γ since α is 2-independent. Take a vertex p in V_1 ; then there exists a unique edge e in E_p such that $\alpha(e) = \beta_1$. Let q be another endpoint of e . Since $\alpha(E_p) \equiv \alpha(E_q) \pmod{\alpha(e) = \beta_1}$ and $f(x_1, \dots, x_n)$ is symmetric, one has that

$f(\alpha(E_p)) \equiv f(\alpha(E_q)) \pmod{\beta_1}$. Furthermore, by taking the common denominator, $\frac{f(\alpha(E_p))}{\prod_{y \in E_p} \alpha(y)} + \frac{f(\alpha(E_q))}{\prod_{z \in E_q} \alpha(z)}$ becomes

$$\frac{f(\alpha(E_p))g_1 + f(\alpha(E_q))g_2}{\beta_1\beta'_2 \cdots \beta'_v}$$

where $g_1, g_2 \in \mathbb{Z}_2[\rho_1, \dots, \rho_k]$. Obviously, $f(\alpha(E_p))g_1 + f(\alpha(E_q))g_2$ can be divided by β_1 . Thus, we can write

$$(2.1) \quad \sum_{p \in V_1} \frac{f(\alpha(E_p))}{\prod_{e \in E_p} \alpha(e)} = \frac{h_1}{\beta_1'' \cdots \beta_{u_1}''}$$

such that each $\beta_i'' \in \text{Hom}(G, \mathbb{Z}_2)$ is not equal to β_1 . Since $\beta_1 \notin \alpha(E_p)$ for $p \in V_\Gamma \setminus V_1$, one has

$$(2.2) \quad \sum_{p \in V_\Gamma \setminus V_1} \frac{f(\alpha(E_p))}{\prod_{e \in E_p} \alpha(e)} = \frac{h_2}{\beta_1''' \cdots \beta_{u_2}'''}$$

such that each $\beta_i''' \in \text{Hom}(G, \mathbb{Z}_2)$ is not equal to β_1 . Combining (2.1) and (2.2), one has

$$\sum_{p \in V_\Gamma} \frac{f(\alpha(E_p))}{\prod_{e \in E_p} \alpha(e)} = \frac{h'}{\beta_2 \cdots \beta_u}.$$

This means that h is divisible by β_1 . □

The following example shows that generally, the restriction of 2-independence of α in Proposition 2.2 is necessary.

Example 1. Figure 1 provides an abstract 1-skeleton of type (3, 6) with four vertices p, q, r, s , which is not 2-independent. However, this abstract 1-skeleton is not

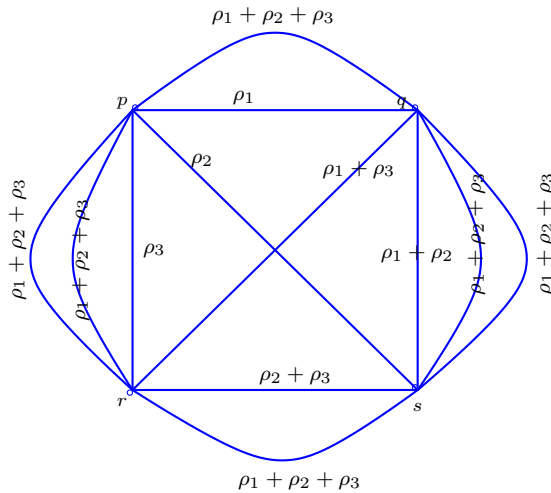


FIGURE 1. An abstract 1-skeleton of type (3, 6) with α no 2-independent.

a G -colored graph of some G -action. This is because if one takes $f(x_1, \dots, x_6) =$

$\sigma_2(x_1, \dots, x_6)\sigma_3(x_1, \dots, x_6)$ with $\deg f = 5$, then by direct computations (cf [L, Claim 2 and Remark 9]),

$$\sum_{u \in \{p, q, r, s\}} \frac{f(\alpha(E_u))}{\prod_{e \in E_u} \alpha(e)} \neq 0$$

so $\sum_{u \in \{p, q, r, s\}} \frac{f(\alpha(E_u))}{\prod_{e \in E_u} \alpha(e)} \notin \mathbb{Z}_2[\rho_1, \rho_2, \rho_3]$. Note that for each u , $\deg \prod_{e \in E_u} \alpha(e) = 6$.

Remark 2. We tried to show that if α is 2-independent, then (Γ, α) is a G -colored graph of some G -action, but failed. Even so, it is extremely tempting to *conjecture* that this is true.

3. FACES

Suppose that (Γ, α) is an abstract 1-skeleton of type (k, n) . Let Γ' be a connected ℓ -valent subgraph of Γ where $0 \leq \ell \leq n$.

We say that $(\Gamma', \alpha|_{\Gamma'})$ is a ℓ -dimensional face of (Γ, α) if there is a subspace K of $\text{Hom}(G, \mathbb{Z}_2)$ such that

- (1) for each vertex $p \in V_{\Gamma'}$, the image $\alpha(E_p|_{\Gamma'})$ spans K , and
- (2) for each edge $e = pq \in E_{\Gamma'}$,

$$\alpha(E_p|_{\Gamma'}) \equiv \alpha(E_q|_{\Gamma'}) \pmod{\alpha(e)}.$$

Obviously, each edge e of (Γ, α) , with the restriction of α to it, is a 1-dimensional face. (Γ, α) is the union of some n -faces, and in particular, (Γ, α) itself is a unique n -face if Γ is connected. Note that each vertex of Γ is a 0-face. The following example shows that generally, for m edges of E_p at a vertex p where $1 < m < n$, there may be no m -face containing the m edges.

Example 2. Figure 2 gives an abstract 1-skeleton (Γ, α) of type $(3, 4)$ such that α is three-independent, but there are three edges at the vertex p colored by ρ_1, ρ_2, ρ_3 , respectively such that they cannot extend to a 3-face.

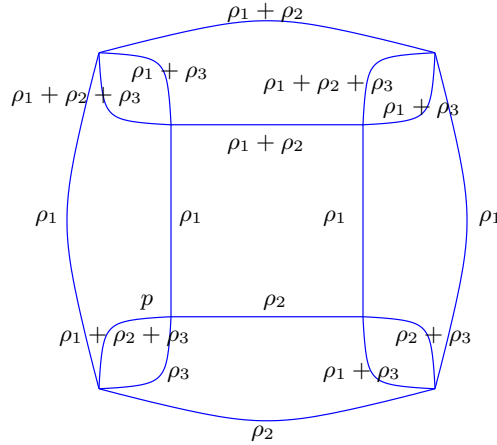


FIGURE 2. An abstract 1-skeleton of type $(3, 4)$ with α three-independent.

The existence of a face of (Γ, α) depends heavily on the l -independence of α . Our result for (Q2) is stated as follows.

Proposition 3.1. *Suppose that (Γ, α) is an abstract 1-skeleton of type (k, n) such that α is l -independent where $1 < l \leq k$. Then any $m(< l)$ edges e_1, \dots, e_m of E_p at $p \in V_\Gamma$ can extend to a unique m -face. In particular, when $k = n$, any $m(\leq n)$ edges of E_p at $p \in V_\Gamma$ can always extend to a unique m -face.*

Proof. It is trivial when $l = 2$. If $l \geq 3$, write $E = \{e_1, \dots, e_m\} \subset E_p$, one then proceeds as follows. One begins with p as being the starting point. Since α is l -independent and $m < l$, the space $K = \text{Span}\{\alpha(e)|e \in E\}$ has dimension m and is different from the space $\text{Span}\{\alpha(e)|e \in E'\}$, where $E' \subset E_p$ contains m edges and $E' \neq E$. Take an edge \bar{e} in E and let q be another endpoint of \bar{e} , one then has that there is a unique subset F of containing m edges in E_q such that $K = \text{Span}\{\alpha(e)|e \in F\}$ and $\alpha(E) \equiv \alpha(F) \pmod{\alpha(\bar{e})}$. One may further carry out this procedure at q as follows: take an edge \bar{e}' in F and let r be another vertex of \bar{e}' , then one finds a unique subset H of E_r of containing m edges such that $K = \text{Span}\{\alpha(e)|e \in H\}$ and $\alpha(F) \equiv \alpha(H) \pmod{\alpha(\bar{e}')}.$ Continuing this procedure, since Γ is assumed to be finite, finally one can obtain a unique connected m -valent subgraph containing e_1, \dots, e_m as desired. \square

Remark 3. As shown in [GZ2, Proposition 2.1.3], the three-independence of α for a GKM graph (Γ, α) determines a unique connection θ . This is also true for our abstract graphs in mod 2 category. However, the three-independence of α cannot determine the existence of faces of dimension more than 2, as is shown in Example 2.

We also can further observe the property of the intersection of faces of (Γ, α) .

Proposition 3.2. *Suppose that (Γ, α) is an abstract 1-skeleton of type (k, n) such that α is l -independent, where $l \leq k$. Then for faces F^{m_1}, \dots, F^{m_s} of dimension less than l in (Γ, α) with $s > 1$, their intersection $F^{m_1} \cap \dots \cap F^{m_s}$ is either empty or the disjoint union of lower-dimensional faces.*

Proof. Suppose that the intersection $F^{m_1} \cap \dots \cap F^{m_s} \neq \emptyset$. Then $F^{m_1} \cap \dots \cap F^{m_s}$ contains at least one vertex (i.e., a 0-face) p of Γ . Suppose $E_p|_{F^{m_1} \cap \dots \cap F^{m_s}}$ contains only m edges in E_p . Then $0 \leq m \leq \min(m_1, \dots, m_s)$ and by Proposition 3.1 the intersection $F^{m_1} \cap \dots \cap F^{m_s}$ must contain a m -face F^m containing the m edges in E_p . If there is also another vertex $q \in F^{m_1} \cap \dots \cap F^{m_s}$ such that $q \notin F^m$, in the above way one may obtain a face $F^{m'}$ containing q .

One claims that $F^{m'}$ is another connected component different from F^m in the intersection $F^{m_1} \cap \dots \cap F^{m_s}$. With no loss of generality, one assumes that $m' \leq m$. If the dimension of $F^{m'}$ is zero (i.e., $F^{m'} = \{q\}$), then obviously the claim holds. If the dimension of $F^{m'}$ is more than zero and if the intersection of $F^{m'}$ and F^m is nonempty, then there is a vertex o in Γ such that $o \in F^{m'} \cap F^m$. Obviously, all edges of $F^{m'}$ with endpoint o cannot belong to F^m . Indeed, otherwise $F^{m'}$ would be a subface of F^m , so $q \in F^m$, which gives a contradiction. Therefore, there is an edge e in $F^{m'}$ with endpoint o such that $e \notin F^m$. Since $F^m \subset F^{m_1} \cap \dots \cap F^{m_s}$ and $F^{m'} \subset F^{m_1} \cap \dots \cap F^{m_s}$, one has that $F^{m_1} \cap \dots \cap F^{m_s}$ contains at least $m+1$ edges at E_o such that these $m+1$ edges extend a $(m+1)$ -face F^{m+1} with F^m as its a subface, and in particular, F^{m+1}

contains the vertex p . Since $F^{m+1} \subset F^{m_1} \cap \dots \cap F^{m_s}$, this means that $F^{m_1} \cap \dots \cap F^{m_s}$ contains $m+1$ edges of E_p . This is a contradiction. \square

The following is an application for the d -connectedness of a graph. Before one discusses this, let us review the notion for the d -connectedness of a graph and the Whitney Theorem (cf [G]).

Let Γ be a graph. A *path* σ with two endpoints a, b in Γ is a subgraph of Γ having as vertices the vertices $v_0 = a, v_1, \dots, v_{l-1}, v_l = b$ of Γ and as edges the edges $v_{i-1}v_i, i = 1, \dots, l$ of Γ . Two paths σ_1 and σ_2 with common endpoints a, b are called *disjoint* provided the intersection $\sigma_1 \cap \sigma_2$ consists of a and b only. A graph Γ is *connected* provided for each pair of its vertices there is a path in Γ having these vertices as endpoints. A graph Γ is *d -connected* provided for every pair of vertices of Γ there exist d pairwise disjoint paths in Γ having these vertices as endpoints.

Theorem 3.3. (Whitney) *A graph Γ with at least $d+1$ vertices is d -connected if and only if every subgraph of Γ obtained by deleting from Γ any $d-1$ or fewer vertices and the edges incident to them is connected.*

Lemma 3.1. *Suppose that (Γ, α) is an abstract 1-skeleton of type (k, n) with Γ connected such that α is at least three independent. Then Γ must be at least 2-connected.*

Proof. If Γ is 1-connected but not 2-connected, then by Whitney Theorem there must be at least an edge $e = pq$ in Γ such that $\Gamma \setminus \{e\}$ is disconnected. Take an edge e' in $E_p \setminus \{e\}$, since α is at least three independent, by Proposition 3.1 there should be a 2-face F containing e and e' . However, this is impossible since $\Gamma \setminus \{e\}$ is disconnected. \square

Proposition 3.4. *Suppose that (Γ, α) is an abstract 1-skeleton of type (k, n) with Γ connected such that α is at least three independent. If the intersection of any two faces of dimension less than 3 in (Γ, α) is either connected or empty, then Γ is n -connected.*

Proof. Suppose that Γ is not n -connected. Then by Whitney Theorem and Lemma 3.1, there is at least one subset S with its cardinality $2 \leq |S| \leq n-1$ of the vertex set V_Γ such that $\Gamma \setminus S$ is disconnected, where $\Gamma \setminus S$ denotes the subgraph of Γ obtained by removing all vertices in S and all edges adjacent to those vertices. Let S_{\min} be *minimal* (i.e., its cardinality is minimal) among subsets S for which $\Gamma \setminus S$ is disconnected. Fix a vertex p of S_{\min} .

Claim 1. Each connected component of $\Gamma \setminus S_{\min}$ contains a vertex adjacent to p .

Take any vertex q of $\Gamma \setminus S_{\min}$. Since S_{\min} is minimal, $\Gamma \setminus (S_{\min} \setminus \{p\})$ is connected; so there is a path in $\Gamma \setminus (S_{\min} \setminus \{p\})$ from q to p . Cutting the last edge from this path produces a path in $\Gamma \setminus S_{\min}$ from q to a vertex adjacent to p . This proves the claim 1.

Let q, q' be vertices of $\Gamma \setminus S_{\min}$ which belong to different connected components of $\Gamma \setminus S_{\min}$ and are adjacent to p . Then one has that

Claim 2. The 2-face determined by the two edges joining p and q, q' contains a vertex in S_{\min} except p .

Indeed, if the 2-face does not contain any vertex in S_{\min} except p , then it gives a path joining q and q' in $\Gamma \setminus S_{\min}$, but this contradicts the assumption that q and q' belong to different connected component of $\Gamma \setminus S_{\min}$.

Now let r be the number of vertices in S_{\min} adjacent to p . Then one has

Claim 3. There are at least $n - r - 1$ two-faces which contain p and some other vertex of S_{\min} not adjacent to p .

Let $N(p)$ be the set of vertices of $\Gamma \setminus S_{\min}$ adjacent to p . The cardinality of $N(p)$ is $n - r$. A connected component of $N(p)$ means the intersection of $N(p)$ with a connected component of $\Gamma \setminus S_{\min}$. By Claim 1, the number of connected components of $N(p)$ is the same as that of connected components of $\Gamma \setminus S_{\min}$. Suppose that a connected component C of $N(p)$ has d vertices, where $1 \leq d \leq n - r - 1$. Choosing one vertex from C and the other one from $N(p) \setminus C$, one gets a 2-face F containing two edges joining p and the chosen two vertices. F contains p and a vertex in S_{\min} except p , say v , by Claim 2. If v is adjacent to p , then the intersection of F and a 1-face joining p and v consists of p, v (i.e., two 0-faces) and hence is disconnected. But this is impossible, so v is not adjacent to p . There are $d(n - r - d)$ such 2-faces and it is easy to see that $d(n - r - d) \geq n - r - 1$.

On the other hand, the number of vertices in $S_{\min} \setminus \{p\}$ which are not adjacent to p is $|S_{\min}| - r - 1 \leq n - r - 2$. This together with Claim 3 implies that there are at least two different 2-faces F_1, F_2 which contain p and some other vertex, say q , of S_{\min} not adjacent to p . Thus, the intersection of two 2-faces F_1, F_2 contains p and q , which are not adjacent to each other. Since α is at least three independent, by Proposition 3.2 the intersection $F_1 \cap F_2$ consists of the disjoint union of some 1-faces and 0-faces. This means that the intersection $F_1 \cap F_2$ is disconnected. This is a contradiction. Therefore $|S_{\min}| \geq n$ and Γ is n -connected. \square

Note. The inverse of Proposition 3.4 is generally untrue. For example, Figure 3 gives an abstract 1-skeleton (Γ, α) of type $(3, 3)$ such that α is three-independent. Obviously, Γ is 3-connected, but a direct observation shows that there are at least two 2-faces in (Γ, α) such that their intersection is disconnected.

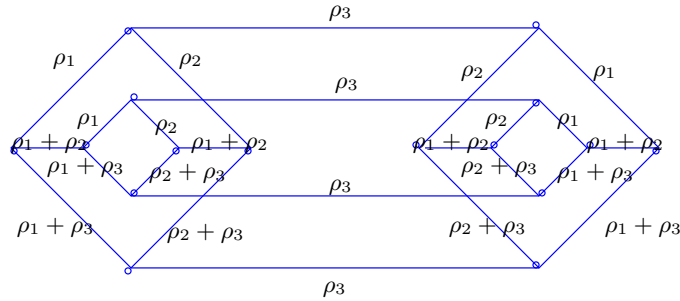


FIGURE 3. An abstract 1-skeleton of type $(3, 3)$

4. ABSTRACT 1-SKELETONS OF TYPE (n, n)

In this section, we are concerned with the case $k = n$. Suppose that (Γ, α) is an abstract 1-skeleton of type (n, n) . In this case, α is n -independent so by Proposition 3.1 any $m(\leq n)$ edges of E_p at $p \in V_\Gamma$ can extend to a unique m -face. Let $\mathcal{F}_{(\Gamma, \alpha)}$ denote the set of all faces of an abstract 1-skeleton (Γ, α) of type (n, n) . We call an $(n - 1)$ -face of $\mathcal{F}_{(\Gamma, \alpha)}$ a *facet* (cf [MMP]).

Lemma 4.1. *Let F^m be a m -face with $m < n$ in $\mathcal{F}_{(\Gamma, \alpha)}$. Then there exist $n - m$ facets F_1, \dots, F_{n-m} in $\mathcal{F}_{(\Gamma, \alpha)}$ such that F^m is a connected component of the intersection $F_1 \cap \dots \cap F_{n-m}$.*

Proof. For a vertex p in F^m , F^m contains m edges of E_p . Write $E_p = \{e_1, \dots, e_n\}$, with no loss one assumes that e_1, \dots, e_m belong to F^m . By Proposition 3.1, any $n - 1$ edges of E_p determine a facet, so there are exactly n facets with p as a vertex. Obviously there exactly are $n - m$ facets F_1, \dots, F_{n-m} with p as a vertex such that each F_i contains these m edges e_1, \dots, e_m , and does not contain at least one edge from $E_p \setminus \{e_1, \dots, e_m\}$. Since any facet of containing e_1, \dots, e_m must contain F^m , one has that each facet F_i contains F^m . The lemma then follows from Proposition 3.2. \square

Every face of (Γ, α) corresponds to a subspace of $\text{Hom}(G, \mathbb{Z}_2)$.

Lemma 4.2. *Suppose that (Γ, α) is an abstract 1-skeleton of type (n, n) . Let Γ' be a connected m -valent subgraph of Γ . Then $(\Gamma', \alpha|_{E_{\Gamma'}})$ is an m -face with $m > 0$ if and only if $\alpha(E_{\Gamma'})$ spans an m -dimensional of $\text{Hom}(G, \mathbb{Z}_2)$.*

Proof. If $(\Gamma', \alpha|_{E_{\Gamma'}})$ is an m -face, then for any two different vertices p_1, p_2 in $V_{\Gamma'}$, $\alpha(E_{\Gamma'}|_{p_1})$ and $\alpha(E_{\Gamma'}|_{p_2})$ span the same m -dimensional vector subspace of $\text{Hom}(G, \mathbb{Z}_2)$, so $\alpha(E_{\Gamma'})$ spans an m -dimensional of $\text{Hom}(G, \mathbb{Z}_2)$.

Conversely, suppose $\alpha(E_{\Gamma'})$ spans a m -dimensional of $\text{Hom}(G, \mathbb{Z}_2)$. Then, since α is n -independent, for any two vertices p_1 and p_2 in $V_{\Gamma'}$, $\text{Span}\alpha(E_{\Gamma'}|_{p_1})$ and $\text{Span}\alpha(E_{\Gamma'}|_{p_2})$ must be the same m -dimensional subspace of $\text{Hom}(G, \mathbb{Z}_2)$. An easy observation also shows that for each edge $e = pq$ in Γ' , $\alpha(E_{\Gamma'}|_p) \equiv \alpha(E_{\Gamma'}|_q) \pmod{\alpha(e)}$. Thus, $(\Gamma', \alpha|_{E_{\Gamma'}})$ is an m -face. \square

As vector spaces over \mathbb{Z}_2 , $\text{Hom}(G, \mathbb{Z}_2)$ and $\text{Hom}(\mathbb{Z}_2, G)$ are dual to each other. By Lemma 4.2, each m -face $F^m = (\Gamma', \alpha|_{E_{\Gamma'}})$ with $m > 0$ of $\mathcal{F}_{(\Gamma, \alpha)}$ actually corresponds to a unique $(n - m)$ -dimensional subspace J of $\text{Hom}(\mathbb{Z}_2, G)$ such that for each $a^* \in J$ and each $a \in \alpha(E_{\Gamma'})$,

$$a^*(a) = 0.$$

Thus, each facet of $\mathcal{F}_{(\Gamma, \alpha)}$ corresponds to a unique nonzero element of $\text{Hom}(\mathbb{Z}_2, G)$. This gives a map

$$\lambda : \overline{\mathcal{F}}_{(\Gamma, \alpha)} \longrightarrow \text{Hom}(\mathbb{Z}_2, G)$$

where $\overline{\mathcal{F}}_{(\Gamma, \alpha)}$ denotes the set of all facets of $\mathcal{F}_{(\Gamma, \alpha)}$. For each vertex p , let F_1, \dots, F_n be n facets with p as a vertex. Then it is easy to see that $\lambda(F_1), \dots, \lambda(F_n)$ are linearly independent in $\text{Hom}(\mathbb{Z}_2, G)$. Given a m -face F^m , by Lemma 4.1 one has that there exist $n - m$ facets F_1, \dots, F_{n-m} in $\mathcal{F}_{(\Gamma, \alpha)}$ such that F^m is a connected component of the

intersection $F_1 \cap \cdots \cap F_{n-m}$. Then F^m corresponds to the $(n-m)$ -dimensional vector space spanned by $\lambda(F_1), \dots, \lambda(F_{n-m})$. Here one calls λ the *characteristic function*. Combining the above arguments, one has

Proposition 4.1. *Suppose that (Γ, α) is an abstract 1-skeleton of type (n, n) . Then α and λ determine each other.*

Remark 4. If (Γ, α) is a colored graph of a small cover M over a simple convex polytope P , then Γ is exactly the 1-skeleton of P , and each facet of $\mathcal{F}_{(\Gamma, \alpha)}$ corresponds to a facet of P . For the notion of a small cover, see [DJ]. Thus, the map λ defined above is actually the characteristic function of the small cover M . Furthermore, combining the GKM theory and the Davis-Januszkiewicz theory for small covers together, we see that the face ring of P over \mathbb{Z}_2 is isomorphic to

$$\{f : V_\Gamma \longrightarrow \mathbb{Z}_2[\rho_1, \dots, \rho_n] \mid f(p) \equiv f(q) \pmod{\alpha(e) \text{ for } e \in E_p \cap E_q}\}.$$

It should be pointed out that an abstract 1-skeleton of type (n, n) is an analogue of a torus graph introduced by Maeda, Masuda and Panov in [MMP]. They have shown that the equivariant cohomology of a torus graph is isomorphic to the face ring of the associated simplicial poset. This is a generalization of the above isomorphism.

Now suppose that (Γ, α) is an abstract 1-skeleton of type (n, n) with Γ connected. Then $\mathcal{F}_{(\Gamma, \alpha)}$ contains a unique n -face (Γ, α) . As an analogue, (Γ, α) has many same properties as a torus graph, which are stated as follows:

- (i) $\mathcal{F}_{(\Gamma, \alpha)}$ forms a simplicial poset of rank n with respect to reversed inclusion with (Γ, α) as smallest element, denoted by $\mathcal{P}_{(\Gamma, \alpha)}$.
- (ii) $\mathcal{P}_{(\Gamma, \alpha)}$ is a face poset of a simplicial complex K if and only if all possible non-empty intersections of facets of $\mathcal{F}_{(\Gamma, \alpha)}$ are connected (see also [MMP, Proposition 5.1]).

As a consequence of Lemma 4.1, Proposition 3.4 and Property (ii), one has that

Corollary 4.2. *If $\mathcal{P}_{(\Gamma, \alpha)}$ is a face poset of a simplicial complex K , then Γ is n -connected.*

We know from [S] or [MMP] that as a simplicial poset, $\mathcal{P}_{(\Gamma, \alpha)}$ determines a regular CW-complex $\mathbb{K}_{\mathcal{P}_{(\Gamma, \alpha)}}$, which is $(n-1)$ -dimensional. By $|\Gamma, \alpha|$ one denotes the underlying space of this cell complex, and one calls it *the geometric realization of (Γ, α)* .

Remark 5. The geometric realization $|\Gamma, \alpha|$ of (Γ, α) has a direct connection with the topology of manifolds. As an interesting topic, the study of $|\Gamma, \alpha|$ has been carried on independently in [BL]. For example, it can be shown that each closed combinatorial manifold can be realizable by $|\Gamma, \alpha|$, and that the geometric realization of an abstract 1-skeleton (Γ, α) of type $(4, 4)$ is a closed 3-manifold if and only if the f -vector of $\mathcal{P}_{(\Gamma, \alpha)}$ satisfies $f_1 = f_0 + f_3$, where f_i denote the number of $a \in \mathcal{P}_{(\Gamma, \alpha)}$ for which the segment $[\hat{0}, a]$ is a boolean algebra of rank $i+1$.

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